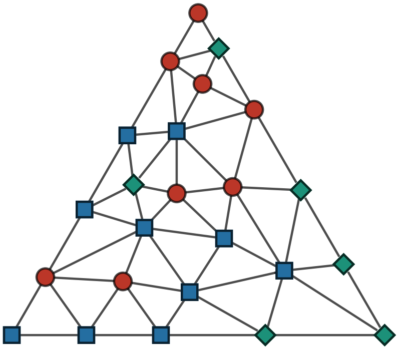
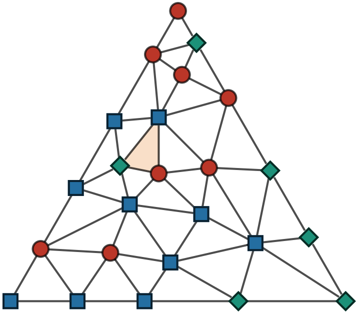
An Initial Activity

Color the vertices on your figure using 3 colors, such that:

1. The three corners are all colored differently
2. The vertices directly between two corners get one of the two colors of those corners
3. The interior vertices can have any color

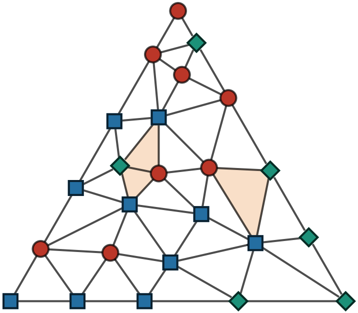


An Initial Activity



- Count the number of triangular cells that have *all three* colors on their corners
- How many did you get?

An Initial Activity



- If you followed the rules, it is guaranteed that you have at least one of these tricolored cells
- In fact, you must have an *odd* number of them
- This fact is guaranteed by a combinatorial result called **Sperner's Lemma**

Introduction

- This example seems trivial and abstract
- But it turns out to have important applications to advanced mathematics and to solving real-world problems
- It also forms unexpected connections between branches of math as diverse as:
  - Combinatorics
  - Graph Theory
  - Real Analysis
  - Topology

Have Your Cake and Eat It:
Applications of Sperner's Lemma to Envy-Free Division Problems

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Department of Applied Mathematics
Student Seminar

September 2, 2026

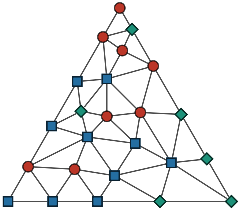
Outline

1. State and prove Sperner's Lemma
2. Explore applications to fair division problems
3. Explore applications to more general fixed point problems

Sperner's Lemma

Background

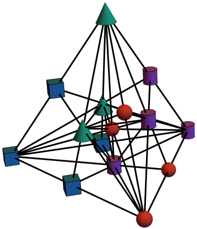
- *Sperner's Lemma* was first published in a 1928 paper by German mathematician Emanuel Sperner
- It concerns the existence of particular colorings of triangulations



- It was originally used to prove several theorems about Topology
- Despite "only" being a *lemma*, it is famous on its own due to its applications in Game Theory and Numerical Analysis

Background

- Sperner's Lemma generalizes to colorings of triangulations in n dimensions

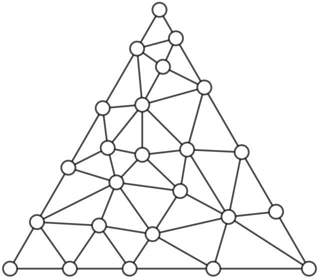


- We will focus on the 2D version since it is easy to visualize and explain
- We will then briefly see its higher-dimension analogs

Background

Definition (Triangulation in 2D)

A *triangulation* of $\triangle ABC$ is a subdivision of $\triangle ABC$ into smaller triangular cells whose union is $\triangle ABC$. Any two cells either meet along one shared edge or do not meet at all.

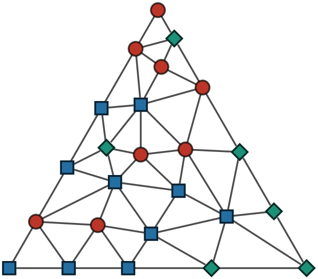


Background

Definition (Sperner Coloring in 2D)

Given a triangulation of $\triangle ABC$, a *Sperner coloring* is an assignment of 3 colors to the vertices such that:

- (a) vertices A, B, and C receive distinct colors
- (b) all vertices on an edge of $\triangle ABC$ receive one of the 2 colors from the endpoints of the edge

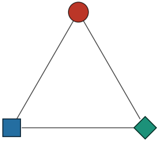


Sperner’s Lemma

Sperner’s Lemma in 2D (1928)

Any Sperner coloring of any triangulation of a triangle must contain at least one fully-labeled cell. (More precisely, there must be an odd number of fully-labeled cells.)

- “Fully-labeled” means that all three colors must appear on the cell’s vertices

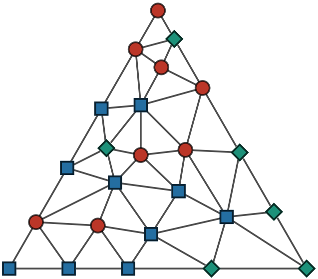


- We will focus on proving the existence of *one* fully-labeled cell for now

Proof by Architecture

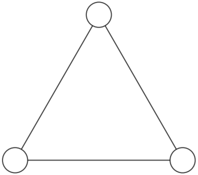
Proof idea:

- Take an arbitrary Sperner coloring of an arbitrary triangulation
- Imagine that the figure represents the floor plan of a house (with cells as rooms and edges as walls)
- Define a “door” on each red/blue wall (i.e. on each edge from a red vertex to a blue vertex)



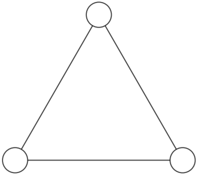
Proof by Architecture

- Question: How many red/blue doors is it possible for a room to have?

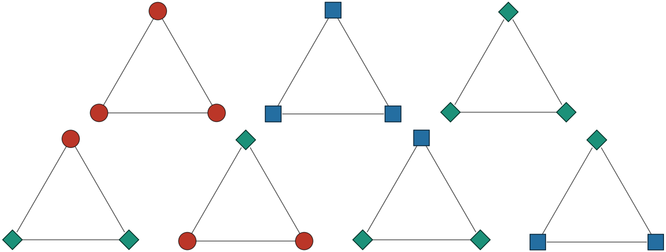


Proof by Architecture

- Question: How many red/blue doors is it possible for a room to have?

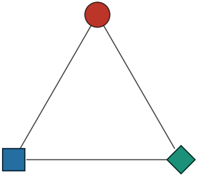


- 0 is possible



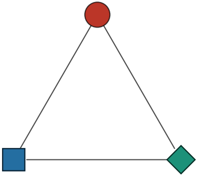
Proof by Architecture

- 1 is possible (and corresponds to a fully-labeled cell)

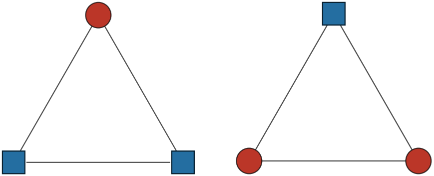


Proof by Architecture

- 1 is possible (and corresponds to a fully-labeled cell)

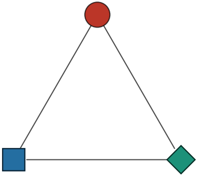


- 2 is possible

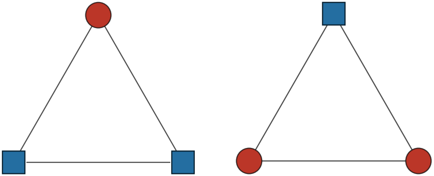


Proof by Architecture

- 1 is possible (and corresponds to a fully-labeled cell)



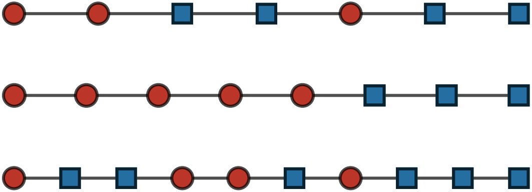
- 2 is possible



- And that's it

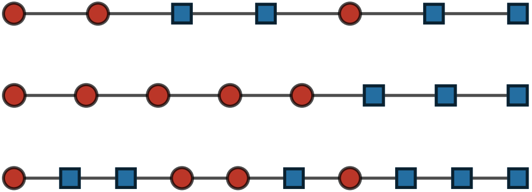
Proof by Architecture

- Question: How many doors could there be on the exterior wall between the red and blue corners?



Proof by Architecture

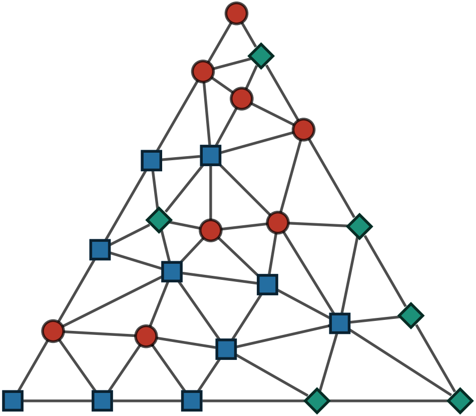
- Question: How many doors could there be on the exterior wall between the red and blue corners?



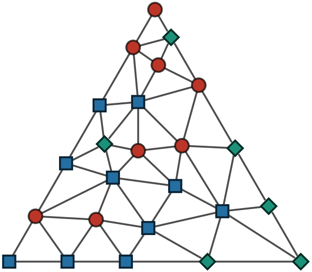
- There must be an *odd* number
- If both ends are opposite colors, the color must swap an *odd* number of times from one end to the other

Proof by Architecture

Complete the proof by thinking about what happens if we keep walking through doors without repetition.



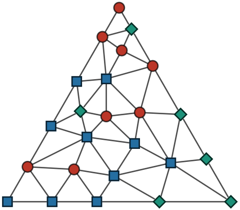
Proof by Graph Theory

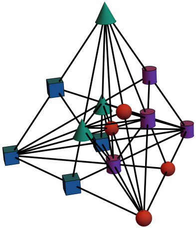


- Sperner's Lemma can also be proved using *graph theory*
- This version of the proof is necessary for generalizing Sperner's Lemma to higher dimensions
- (Ask about the details later if you like)

Generalizing the Result

- We just saw Sperner's Lemma in 2D, which applies to 3-colorings of triangles
- The result can be generalized to higher (or lower) dimensions
- It applies to the higher (or lower) dimension equivalents of "triangles", which are a type of shape called a *simplex*





Generalizing the Result

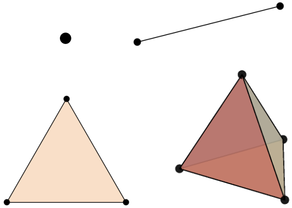
Definition (n-Simplex)

The n-simplex, denoted Δ^n, is the convex hull of the standard basis vectors in $\mathbb{R}^{n+1}$

$$\Delta^n := \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=1}^{n+1} x_i = 1 \text{ and } x_i \geq 0,\ \forall i = 1, \ldots, n+1 \right\}$$

The first few simplices include:

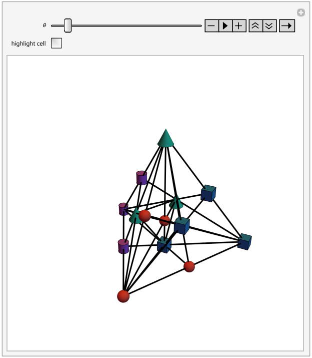
- Point ($n = 0$)
- Line segment ($n = 1$)
- Triangle ($n = 2$)
- Tetrahedron ($n = 3$)



Generalizing the Result

Sperner's Lemma (1928)

Any Sperner coloring of any triangulation of an n-simplex must contain at least one fully-labeled cell. (More precisely, there must be an odd number of fully-labeled cells.)



Envy-Free Division Problems

Envy-Free Cake Division

- Consider a group of friends who want to share a cake
- The cake will be cut, and each piece will be distributed to one of the friends (no waste)

Envy-Free Cake Division

- This is complicated by the fact that the cake is *non-homogeneous*
- It cannot necessarily be divided into pieces that are all identical to each other

Envy-Free Cake Division

- It is further complicated by the fact that each player has their own personal preferences
- Players may disagree about which is the “best” piece in any division

Envy-Free Cake Division

- Our goal is to find an **envy-free** division of the cake
- This means deciding:
  a. how to cut the cake into pieces
  b. which player to give each piece to
- Being “envy-free” means that no player should (strictly) prefer anyone else’s piece to their own

Envy-Free Cake Division

- *Envy-Free Cake Division* is a famous problem from Game Theory
- The example of a cake is obviously silly and inconsequential
- But fair division problems like this arise all the time in the real world
- The “cake” could be anything (resources, job responsibilities, etc.) that can be continuously divided

Envy-Free Cake Division

- If there are only two players, the problem is easy
- A simple protocol for two-player envy-free cake division is:
  1. One player divides the cake into two slices they consider to be equal
  2. The other player decides who gets which slice
- This guarantees that both players like their slice at least as much as the other slice

Envy-Free Cake Division

- For more than two players it's not obvious that an envy-free division is even guaranteed to be possible
- Surprisingly, no matter what the cake looks like or what the players' preferences are, an envy-free division can *always* be found (subject to reasonable assumptions)

Envy-Free Cake Division

Envy-Free Cake Division Theorem (F. Simmons 1980)

There exists an envy-free cake division for n players if:

a. *All players are hungry (all players would pick any nonempty piece over an empty piece).*
b. *Preference sets are closed (if a player prefers the same piece in every cut of a convergent sequence of cuts, they also prefer that piece in the limiting cut).*

Envy-Free Cake Division

- Several proofs and division algorithms exist
- We will see one that uses Sperner's Lemma
- This can be generalized to any number of players, but for now we will stick with 3 players: **Alice**, **Bob**, and **Charlie**

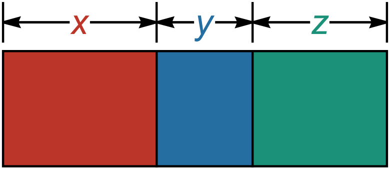
The Geometry of the Cuts

- First, we simplify the problem by restricting its scope
- For a fully 3D (or even 2D) cake, many different and intricate cuts are possible

The Geometry of the Cuts

- We will assume that all cuts are planar and parallel to each other
- Then each set of cuts can be fully described by just specifying, in order, how *wide* each slice is

The Geometry of the Cuts



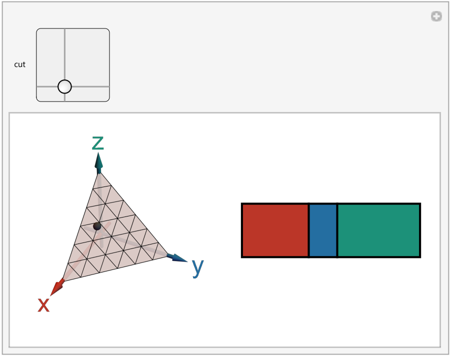
- This effectively reduces the cake to one dimension
- Without loss of generality, say the cake has total width 1
- Call the widths of the three slices x, y, and z, respectively
- Assuming the slices are arranged in a consistent order, fully specifying a division of the cake just means specifying three numbers: x, y, and z

The Geometry of the Cuts

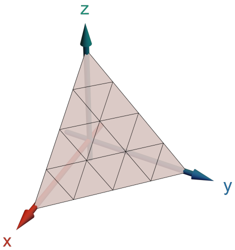
- In order to be a valid cut, the triple (x, y, z) must satisfy:
  - $x \geq 0$, $y \geq 0$, $z \geq 0$
  - $x + y + z = 1$

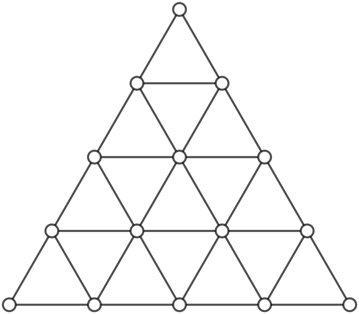
The Geometry of the Cuts

- In order to be a valid cut, the triple (x, y, z) must satisfy:
  - $x \geq 0$, $y \geq 0$, $z \geq 0$
  - $x + y + z = 1$
- Then each cut corresponds to a point inside the *2-simplex* in $\mathbb{R}^3$



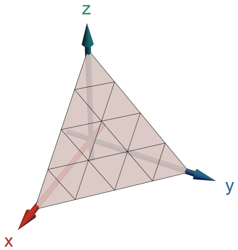
Preference Surveys

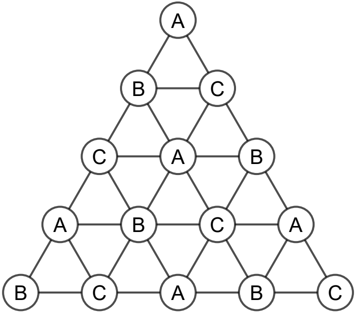




- Draw a triangular grid on the 2-simplex
- Each vertex of the grid corresponds to one possible cut

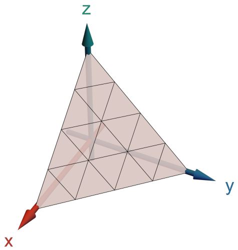
Preference Surveys

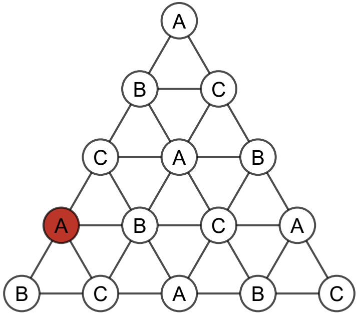




- Assign each vertex to one of the three players (**A**lice, **B**ob, or **C**harlie)
- Ensure that each triangular cell is surrounded by all three players

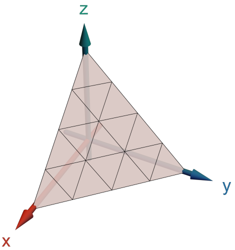
Preference Surveys

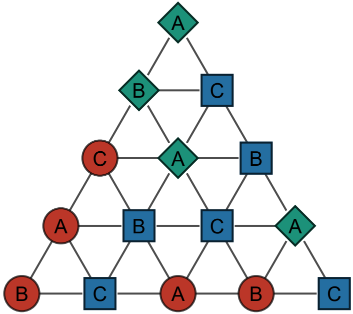




- For each vertex, ask the corresponding player which piece they would prefer from the cut represented by that vertex
- e.g. Alice would prefer slice 1 if the cut were $(\frac{3}{4}, 0, \frac{1}{4})$
- Color the vertex according to that player's choice

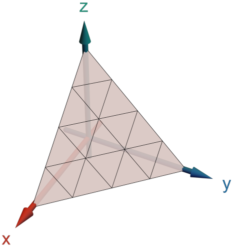
Lemma Drizzle

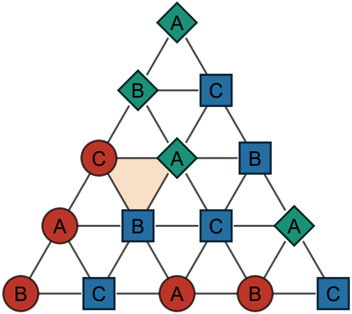




- Claim: Doing so for all points on the grid must result in a Sperner coloring
- Why?

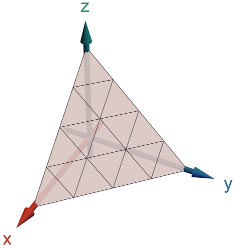
Lemma Drizzle

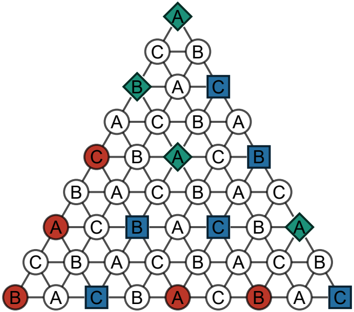




- Then by Sperner's Lemma there must be at least one fully-labeled cell
- Its corners represent three "similar" cuts for which all three players prefer a different piece

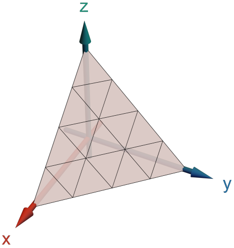
Lemma Drizzle

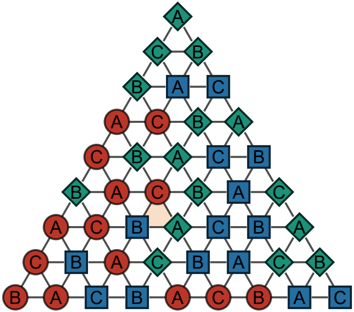




- If none of these cuts is agreeable to all three players, the process can be repeated on a finer mesh
- The triangular grid can be subdivided while preserving the original choices

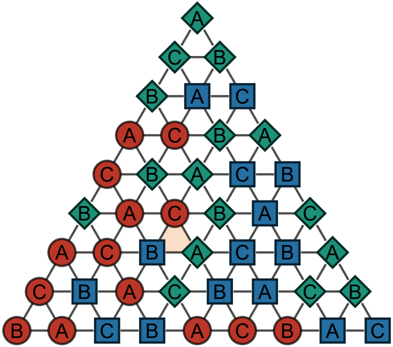
Lemma Drizzle





- The new mesh must still contain a fully-labeled cell
- But now the cell is smaller, and the cuts corresponding to its corners are even more similar

Lemma Drizzle



- This process can be used to algorithmically compute an ε-approximation of an envy-free division
- Taking limits and applying the *closed preference set* assumption, we can use it to show the existence of an exact envy-free division
- (Ask about the details later if you like)

Fixed Point Results

Fixed Point Theorems

- Many important results from throughout applied mathematics rely on showing existence of **fixed points**

$$f(x^*) = x^*$$

- Sperner's Lemma can be used to show some of these

Fixed Point Theorems

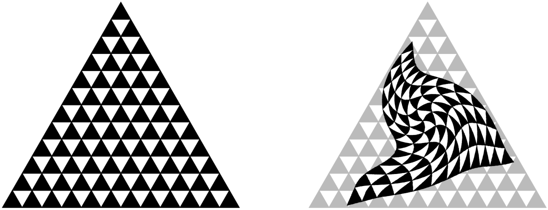
- In Computational Mathematics, Sperner's Lemma can be used to develop some root-finding algorithms
- The foundational result from Game Theory, the existence of *Nash equilibria*, can be proved using Sperner's Lemma
- We will see how Sperner's Lemma can be used to prove a famous result from Topology called the **Brouwer Fixed Point Theorem**

The Brouwer Fixed Point Theorem

Brouwer Fixed Point Theorem (1911)

Let D be any convex, compact subset of $\mathbb{R}^n$ and let $\mathbf{f}$ be any continuous function from D to itself. Then $\mathbf{f}$ must have at least one fixed point.

- $\mathbf{f}$ is a continuous function that moves points in a region D to other points in that same region
- There must always be at least one point that remains stationary during the transformation, i.e. $\mathbf{x} \in D$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{x}$



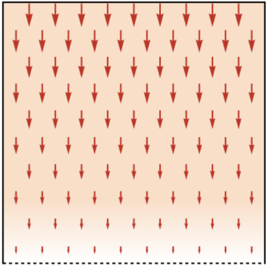
The Brouwer Fixed Point Theorem

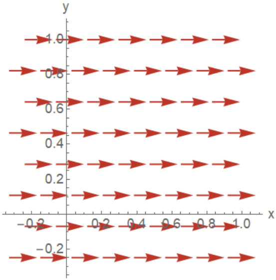
Some famous intuitive implications of the Brouwer Fixed Point Theorem include:

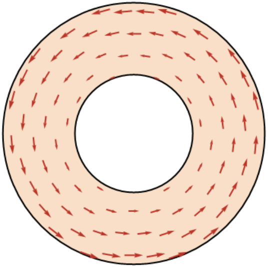
- Crumpling a paper and placing it on top of a copy always leaves at least one point above its original location
- Any map of your current location can have a valid “you are here” dot drawn on it
- Stirring a cup of coffee always stirs at least one molecule back to where it began

Necessary Conditions

- *Compactness* (in Euclidean space) means that the set must be closed and bounded
- *Convexity* requires that the set not contain any holes
- Functions on non-closed, unbounded, or nonconvex sets need not have fixed points

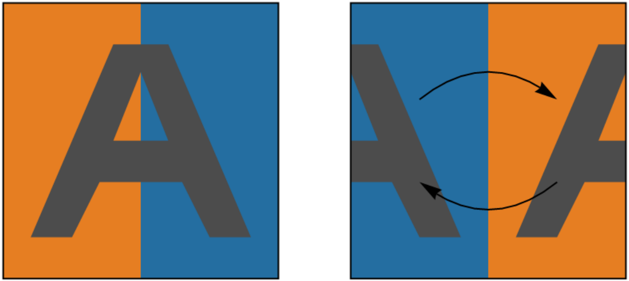






Necessary Conditions

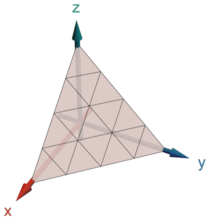
- *Continuity* requires that “nearby” points remain “nearby” after applying the function
- Discontinuous functions need not have fixed points

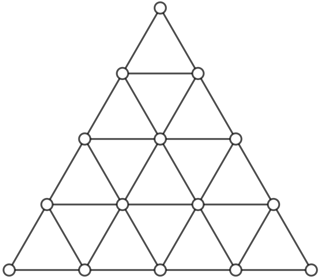


Proof of the Brouwer Fixed Point Theorem

- Many proofs of the Brouwer Fixed Point Theorem exist
- We will see one that applies Sperner's Lemma
- We will focus on the 2D case, but this generalizes to any number of dimensions

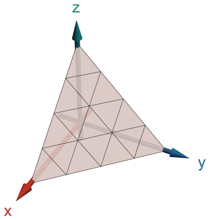
Proof of the Brouwer Fixed Point Theorem

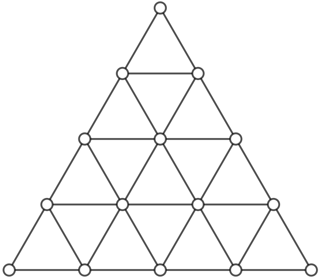




- It suffices to show that the Brouwer Fixed Point Theorem works when D is a simplex
- Δ^n is compact, convex, and homeomorphic to any other closed, compact set with dimension n

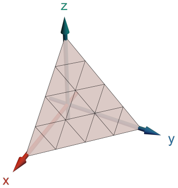
Proof of the Brouwer Fixed Point Theorem





- Let $\mathbf{f} = (f_1, f_2, f_3)$ be any continuous function that maps the 2-simplex to itself
- We again begin by drawing a triangular grid of points on the 2-simplex

Proof of the Brouwer Fixed Point Theorem

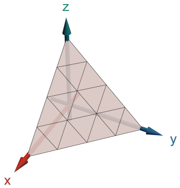


- Observation: For any $\mathbf{x} = (x_1, x_2, x_3) \in \Delta^2$, there must be at least one coordinate that is *not* increased by $\mathbf{f}$, i.e.

$$\text{either} \quad f_1(\mathbf{x}) \leq x_1 \quad \text{or} \quad f_2(\mathbf{x}) \leq x_2 \quad \text{or} \quad f_3(\mathbf{x}) \leq x_3$$

- Why?

Proof of the Brouwer Fixed Point Theorem



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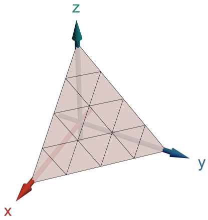
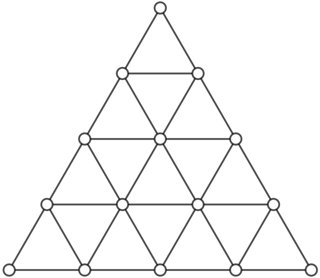
- Why?
- Otherwise all three coordinates would increase, in which case

$$f_1(\mathbf{x}) + f_2(\mathbf{x}) + f_3(\mathbf{x}) > x_1 + x_2 + x_3 = 1$$

which would place $\mathbf{f}(\mathbf{x})$ outside Δ^2

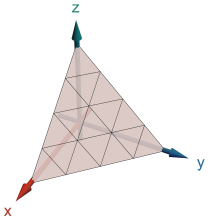
Proof of the Brouwer Fixed Point Theorem

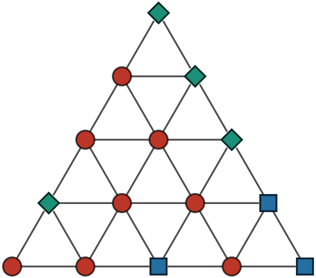




- The idea is to color the points on the simplex according to which coordinate does *not* increase
- e.g. red if x does not increase, blue if y does not increase, or green if z does not increase

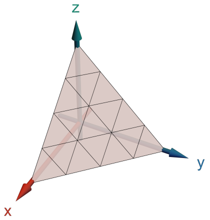
Proof of the Brouwer Fixed Point Theorem

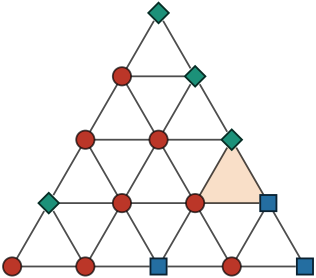




- Corners and edges place restrictions on which coordinates might increase
- e.g. On the x-axis, x cannot increase, so $(1, 0, 0)$ may be red
- e.g. On the yz-plane, y and z cannot *both* increase, so these points may be blue or green

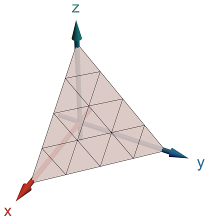
Proof of the Brouwer Fixed Point Theorem

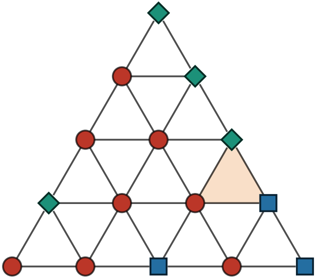




- Then this procedure results in a Sperner coloring
- By Sperner's Lemma, there must be at least one fully-labeled cell
- The corners of this cell represent points for which x, y, and z do not increase (which is a fixed point)

Proof of the Brouwer Fixed Point Theorem





- Using a sufficiently fine mesh provides an ε-approximation of a fixed point
- Taking limits and applying the continuity of $\mathbf{f}$ shows the existence of an exact fixed point
- (Ask about the details later if you like)

Conclusions

Why I Like Sperner's Lemma

Like many results that involve Graph Theory, Sperner's Lemma:

- looks like a children's game
- leads to some pretty pictures
- comes from very simple counting and parity arguments
- leads to some incredibly deep results and useful applications
- shows unexpected connections between discrete and continuous mathematics

Where to Go from Here

If this topic interests you, some classes you might want to take include:

- MAD 3105: Discrete Mathematics II
- Graph Theory(?)
- MAA 4102: Introduction to Real Analysis
- MTG 4302: Elements of Topology I

Thank you!

Recommended Reference:

F. E. Su. Rental Harmony: Sperner's Lemma in Fair Division. *The American Mathematical Monthly*, 106(10):930–942, 1999.

A Bonus Fair Division Problem

Rental Harmony

- Consider a closely-related fair division problem called the **rental harmony problem**
- An apartment with n bedrooms is shared by n tenants
- Not all rooms are equally desirable, and not everyone agrees on the quality of each room

Rental Harmony

- Unlike the cake, the rooms cannot be arbitrarily subdivided
- However, the *rent* can be
- We want to find:
  - an assignment of rooms to tenants
  - a price for each room (as a fraction of the shared rent)

Rental Harmony

- We want the assignment to be *envy-free* in that no tenant should strictly prefer another tenant's room/price to their own
- It turns out we can always find one, subject to some light assumptions

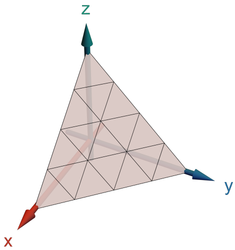
Rental Harmony

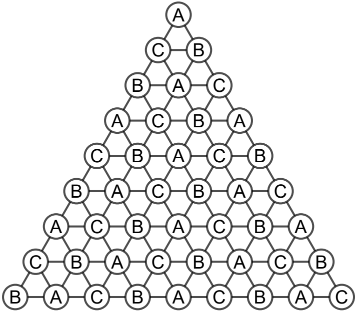
Rental Harmony Theorem (F. E. Su 1999)

Suppose n tenants seek to assign each person to a room and a fraction of the total rent. There exists an envy-free assignment of rooms and rents if the following assumptions hold:

a. *All rooms are acceptable (for any rental division, each person finds at least one room acceptable).*
b. *Tenants are cheap (everyone prefers any free room to any non-free room).*
c. *Preference sets are closed (if a tenant prefers the same room in every division of a convergent sequence of rental divisions, they also prefer that room in the limiting division).*

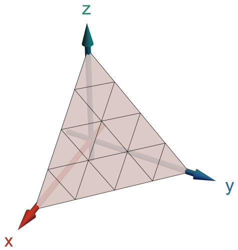
Modifying Sperner's Lemma

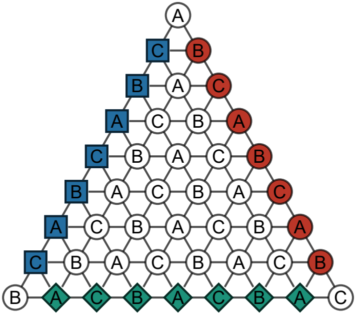




- We'll examine the case of 3 tenants
- Like with the cake division, rental divisions correspond to points in the 2-simplex

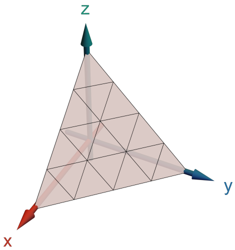
Modifying Sperner's Lemma

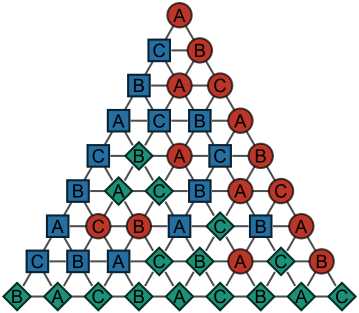




- Like before, color each grid point according to which room the corresponding tenant prefers
- Edges correspond to one room being free of charge, which is always preferred

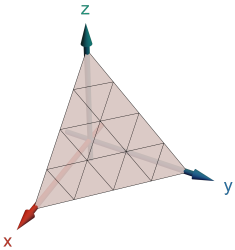
Modifying Sperner's Lemma

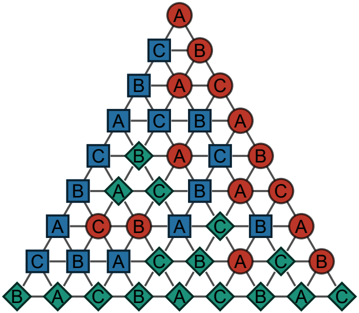




- There is a problem: the assumptions on rental divisions do not generally lead to a Sperner coloring
- The three corners do not necessarily receive three different colors

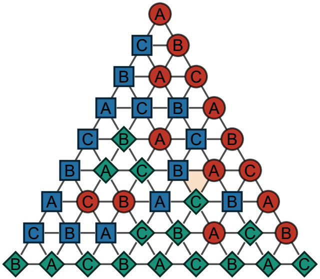
Modifying Sperner's Lemma





- However, we can modify the earlier argument to come up with a slight variation of Sperner's Lemma
- Explain why the new coloring is still guaranteed to result in one fully-labeled cell

Modifying Sperner's Lemma



- Like with the cake division, this method can algorithmically compute an ε-approximation of an envy-free rental division
- Limits and closed preference sets show the existence of an exact envy-free rental division
- The same result holds for some more relaxed assumptions regarding free rooms